Thus, we arrive at the problem studied above.

It is obvious that the procedure described can be applied also to the optimal control problems for the forced motions of elastic shells.

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ON THE REISSNER-NAGHDI ELASTICITY RELATIONSHIPS

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Shell theory equations are constructed by the method in [1] to the accuracy of quantities of the order of h_*^{2+k}, where $k = 0$ for $0 \leqslant t \leqslant 1/2$ and $k = 2-4t$ for $1/2 \leqslant t < 1$ (h_* is the relative semithickness of the shell and t is the index of the state of stress variation). Without being within the framework of the Love-type theory, the equations obtained are compared with the Reissner-Naghdi equations [2, 3] in which the transverse shear is taken into account, and it is shown that from the asymptotic viewpoint these latter are inconsistent. It is also shown that if the shell resists shear weakly, then from the asymptotic viewpoint the Reissner-Naghdi theory is completely well founded.

The three-dimensional equations of elasticity theory are reduced to two-dimensional equations in [1] by using an asymptotic method, i.e. all members of the same order relative to the small parameter h_* are taken into account at each stage of the calculations. It has been shown that without going outside the framework of the ordinary concepts of the Love-type theory of shells (in particular, without taking account of transverse shear), the shell theory equations can be constructed to the accuracy of quantities of the order of h_*^{2-2t}, but it is impossible to exceed this limit without a qualitative complication in the theory.

1. To construct a shell theory to the accuracy of quantities of the order of h_*^{2+k} ($k = 0$ for $t \leqslant 1/2$ and $k = 2-4t$ for $1/2 < t < 1$) let us use the asymptotic representation of the quantities in three-dimensional elasticity theory used in [1].

The terminology and notation used henceforth correspond to that used in [1, 4].

Let us take the equations of three-dimensional elasticity theory referred to a tri-orthogonal coordinate system $(\alpha_1, \alpha_2, \alpha_3)$ as initial equations. The curvilinear coordinates α_1 and α_2 coincide with the lines of curvature of the middle surface, while the lines α_3 are orthogonal thereto.

Let us tranform the initial equations. We introduce the nonsymmetric stress tensor τ_{ik} related to the symmetric tensor σ_{ik} as follows:

$$\tau_i = \left(1 + \frac{\alpha_3}{R_j}\right)\sigma_i, \qquad \tau_{ij} = \left(1 + \frac{\alpha_3}{R_i}\right)\sigma_{ij}$$
$$\tau_{i3} = \left(1 + \frac{\alpha_3}{R_j}\right)\sigma_{i3}, \qquad \tau_3 = \left(1 + \frac{\alpha_3}{R_1}\right)\left(1 + \frac{\alpha_3}{R_2}\right)\sigma_{33} \tag{1.1}$$

and also make the following change of variables:

$$\alpha_i = R\lambda^{-p}\xi_i, \ \alpha_3 = R\lambda^{-l}\zeta, \ h = R\lambda^{-l}, \ p/l = t \tag{1.2}$$

This is customary for the asymptotic method of stretching the scale along the coordinate lines. The coordinates ξ and ζ are chosen in such a way that differentiation with respect to them do not result in any substantial magnification of the desired functions. We take the following asymptotic representation for the stresses and displacements

$$\tau_i = \lambda^l\tau_i^*, \quad \tau_{ij} = \lambda^l\tau_{ij}^*, \quad \tau_{i3} = \lambda^p\tau_{i3}^*$$
$$\tau_3 = \lambda^c\tau_3^* \quad v_i = \lambda^{l-p}v_i^*, \quad v_3 = \lambda^{l-c}v_3^*$$
$$c = \begin{cases} 0, & l \geqslant 2p \\ -l + 2p, & l < 2p \end{cases} \tag{1.3}$$

where all the quantities with the asterisk are of the same order. We write the three-dimensional elasticity theory equations with (1.1)–(1.3) taken into account.

The equilibrium equations are

$$L_i^* + \frac{1}{a_i}\frac{\partial}{\partial\zeta}(a_i^2\tau_{i3}^*) + \lambda^{-l-p}Ra_ia_jq_i = 0$$
$$-\lambda^{-c}L^* + \lambda^{-l+2p-c}F^* + \frac{\partial\tau_3^*}{\partial\zeta} + \lambda^{-l-c}Ra_1a_2q_3 = 0 \tag{1.4}$$

The stress-displacement formulas are

$$\frac{E}{R}a_je_i^* = a_i\tau_i^* - \nu a_j\tau_j^* - \nu\lambda^{-l+c}\tau_3^*$$
$$\frac{E}{R}a_1a_2\frac{\partial v_3^*}{\partial\zeta} = \lambda^{-2l+2c}\tau_3^* - \nu\lambda^{-l+c}(a_1\tau_1^* + a_2\tau_2^*)$$
$$\frac{E}{R}(a_im_i^* + a_jm_j^*) = 2(1+\nu)a_j\tau_{ij}^*$$
$$\frac{E}{R}\left(a_ia_j\frac{\partial v_i^*}{\partial\zeta} + \lambda^{-l+2p-c}a_jg_i^*\right) = 2(1+\nu)\lambda^{-2l+2p}\tau_{i3}^* \tag{1.5}$$

The conditions on the face surfaces of the shell are

$$\left.\frac{\tau_{i3}^*}{a_j}\right|_{\zeta=\pm1} = \pm q_i^{\pm}, \quad \left.\frac{\tau_3^*}{a_1a_2}\right|_{\zeta=\pm1} = \pm\lambda^{-c}q_3^{\pm} \tag{1.6}$$

The following notation is used:

$$L_i{*} = \frac{1}{A_i}\frac{\partial \tau_i{*}}{\partial \xi_i} + \frac{1}{A_j}\frac{\partial \tau_{ij}{*}}{\partial \xi_j} + \lambda^{-p}\frac{R}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}(\tau_i{*} - \tau_j{*}) + \tag{1.7}$$

$$\lambda^{-p}\frac{R}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}(\tau_{ij}{*} + \tau_{ji}{*}), \qquad L{*} = R\left(\frac{\tau_1{*}}{R_1} + \frac{\tau_2{*}}{R_2}\right)$$

$$F{*} = \frac{1}{A_1}\frac{\partial \tau_{13}{*}}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \tau_{23}{*}}{\partial \xi_2} + \lambda^{-p}\frac{R}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\tau_{13}{*} + \lambda^{-p}\frac{R}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\tau_{23}{*}$$

$$e_i{*} = \frac{1}{A_i}\frac{\partial v_i{*}}{\partial \xi_i} + \lambda^{-p}\frac{R}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}v_j{*} + \lambda^{-c}R\frac{v_3{*}}{R_i} \tag{1.8}$$

$$m_i{*} = \frac{1}{A_j}\frac{\partial v_i{*}}{\partial \xi_j} - \lambda^{-p}\frac{R}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}v_j{*}$$

$$g_i{*} = \frac{1}{A_i}\frac{\partial v_3{*}}{\partial \xi_i} - \lambda^{-2p+c}R\frac{v_i{*}}{R_i}, \qquad a_i = 1 + \lambda^{-l}\zeta\frac{R}{R_i}$$

Here and henceforth, each equality containing the subscripts i and j should be considered as two: one equation is obtained for $i = 1$ $(j = 2)$, and another for $i = 2$ $(j = 1)$.

Proceeding exactly as in [1], but retaining more terms, the following expansions of the desired quantities can be obtained in ζ from (1.4) – (1.6)

$$v_3{*} = v_3^{(0)} + \lambda^{-l+c}\zeta v_3^{(1)} + \lambda^{-2l+2p}\zeta^2 v_3^{(2)} \tag{1.9}$$

$$v_i{*} = v_i^{(0)} + \lambda^{-l+2p-c}\zeta v_i^{(1)} + \lambda^{-2l+2p}\zeta^2 v_i^{(2)} + \lambda^{-3l+4p-c}\zeta^3 v_i^{(3)}$$

$$\tau_i{*} = \tau_i^{(0)} + \lambda^{-l+2p-c}\zeta\tau_i^{(1)} + \lambda^{-2l-2p}\zeta^2\tau_i^{(2)} + \lambda^{-3l+4p-c}\zeta^3\tau_i^{(3)}$$

$$\tau_{ij}{*} = \tau_{ij}^{(0)} + \lambda^{-l+2p-c}\zeta\tau_{ij}^{(1)} + \lambda^{-2l+2p}\zeta^2\tau_{ij}^{(2)} + \lambda^{-3l+4p-c}\zeta^3\tau_{ij}^{(3)}$$

$$\tau_{i3}{*} = \tau_{i3}^{(0)} + \zeta\tau_{i3}^{(1)} + \lambda^{-l+2p-c}\zeta^2\tau_{i3}^{(2)} + \lambda^{-2l+2p}\zeta^3\tau_{i3}^{(3)} + \lambda^{-3l+4p-c}\zeta^4\tau_{i3}^{(4)}$$

$$\tau_3{*} = \tau_3^{(0)} + \zeta\tau_3^{(1)} + \lambda^{-l+2p-c}\zeta^2\tau_3^{(2)} + \lambda^{-2l+4p-2c}\zeta^3\tau_3^{(3)} +$$

$$\lambda^{-3l+4p-c}\zeta^4\tau_3^{(4)} + \lambda^{-4l+6p-2c}\zeta^5\tau_3^{(5)}$$

In these formulas $v_3^{(0)}$, $v_3^{(1)}$, . . ., $\tau_3^{(5)}$ are quantities of the same order, for which the following equations hold:

$$v_i^{(n)} = -\frac{1}{n}g_i^{(n-1)} + \lambda^{-c}\frac{R}{nR_i}k_n{}^3 g_i^{(n-2)} + \frac{2(1+\nu)R}{nE}r_n{}^1\tau_{i3}^{(n-1)} \tag{1.10}$$

$$(n = 1, 2, 3)$$

$$v_3^{(n)} = -\frac{\nu R}{nE}(\tau_i^{(n-1)} + \tau_j^{(n-1)}) \qquad (n = 1, 2)$$

$$\tau_i^{(n)} = \frac{E}{R(1-\nu^2)}\left[e_i^{(n)} + \nu e_j^{(n)} + s_n{}^1 k_n{}^3 d_n{}^2 R\left(\frac{1}{R_j} - \frac{1}{R_i}\right)e^{(n-1)}\right] +$$

$$\frac{\nu}{1-\nu}\frac{r_n{}^0 s_n{}^1}{d_n{}^1}\tau_3^{(n)} \qquad (n = 0, \ldots, 3)$$

$$\tau_{ij}^{(n)} = \frac{E}{2R(1+\nu)}\left[m_i^{(n)} + m_j^{(n)} + s_n{}^1 k_n{}^3 d_n{}^2 R\left(\frac{1}{R_i} - \frac{1}{R_j}\right)m_i^{(n-1)}\right]$$

$$(n = 0, \ldots, 3)$$

$$\tau_3^{(n)} = \frac{d_n{}^1}{n}\left(s_n{}^3 d_n{}^2 d_n{}^4 \delta_n{}^5 L^{(n-1)} - \frac{r_n{}^1}{s_n{}^1}F^{(n-1)} - \lambda^{-l}\delta^1{}_n R q_3\right) \quad (n = 1, \ldots, 5)$$

$$\tau_{i3}^{(n)} = -\frac{1}{n}L^{(n-1)} - \frac{\lambda^{-c}}{12}\frac{R}{R_i}\delta_n{}^3 L_i^{(1)} - \delta_n{}^1\left(\lambda^{-l}\frac{2R}{R_i}\tau_{i3}^{(0)} + \lambda^{-l-p}Rq_i\right)$$

$$(n = 1, \ldots, 4)$$

$$\tau_{i3}^{(0)} + \lambda^{-l+2p-c}\tau_{i3}^{(1)} + \lambda^{-3l+4p-c}\tau_{i3}^{(4)} = \frac{\lambda^{-p}}{2} \times \tag{1.11}$$

$$\left[q_i^+ - q_i^- + \lambda^{-l}\frac{R}{R_j}(q_i^+ + q_i^-)\right]$$

$$\tau_{i3}^{(1)} + \lambda^{-2l+2p}\tau_{i3}^{(3)} = \frac{\lambda^{-p}}{2}\left[q_i^+ + q_i^- + \lambda^{-l}\frac{R}{R_j}(q_i^+ - q_i^-)\right]$$

$$\tau_3^{(0)} + \lambda^{-l+2p-c}\tau_3^{(2)} + \lambda^{-3l+4p-c}\tau_3^{(4)} = \frac{\lambda^{-c}}{2} \times$$

$$\left[q_3^+ - q_3^- + \lambda^{-l}R\left(\frac{1}{R_i} + \frac{1}{R_j}\right)(q_3^+ + q_3^-)\right]$$

$$\tau_3^{(1)} + \lambda^{-2l+4p-2c}\tau_3^{(3)} + \lambda^{-4l+6p-2c}\tau_3^{(5)} = \frac{\lambda^{-c}}{2} \times$$

$$\left[q_3^+ + q_3^- + \lambda^{-l}R\left(\frac{1}{R_i} + \frac{1}{R_j}\right)(q_3^+ - q_3^-)\right]$$

The following notation has been used here:

$$k_n^{\ n} = 0, \quad s_n^{\ n} = \lambda^{-2p+c}, \quad r_n^{\ n} = -\lambda^{-l+c}, \quad \delta_n^{\ n} = 1, \quad d_n^{\ n} = \lambda^{-c}$$
$$k_n^{\ m} = s_n^{\ m} = r_n^{\ m} = d_n^{\ m} = 1, \quad \delta_n^{\ m} = 0 \quad (n \neq m)$$

The quantities with negative superscripts should be assumed zero. The notation $e_i^{(0)}$, $e_i^{(2)}$, $g_i^{(0)}$, $m_i^{(k)}$, $L_i^{(k)}$, $L^{(k)}$ and $F^{(m)}$ ($k = 0, \ldots, 3$; $m = 0, \ldots, 4$) is obtained from (1.7) and (1.8) in which the asterisk must be replaced by the superscripts $0, 2, k$ or m, respectively. The following formulas hold for the remaining quantities:

$$e_i^{(1)} = \frac{1}{A_i}\frac{\partial v_i^{(1)}}{\partial \xi_i} + \lambda^{-p}\frac{R}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}v_j^{(1)} + \lambda^{-2p+c}R\frac{v_3^{(1)}}{R_i}$$

$$g_i^{(1)} = \frac{1}{A_i}\frac{\partial v_3^{(1)}}{\partial \xi_i} - \lambda^{-c}R\frac{v_i^{(1)}}{R_i}$$

$$e_i^{(3)} = \frac{1}{A_i}\frac{\partial v_i^{(3)}}{\partial \xi_i} + \lambda^{-p}\frac{R}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}v_j^{(3)}, \qquad g_i^{(2)} = \frac{1}{A_i}\frac{\partial v_3^{(2)}}{\partial \xi_i}$$

The system (1.10), (1.11) contains 43 unknowns and as many equations. This system can be transformed by quantities used in shell theory. As a result of manipulations, we obtain formulas analogous to those Naghdi proposed.

The displacements of the middle surface, the stress resultants and moments are defined as follows by using (1.1), (1.3), (1.9):

$$u = \lambda^{l-p}v_1^{(0)}, \quad v = \lambda^{l-p}v_2^{(0)}, \quad w = -\lambda^{l-c}v_3^{(0)} \tag{1.12}$$

$$T_i = \int_{-h}^{h}\tau_i d\alpha_3 = 2h\lambda^l\left(\tau_i^{(0)} + \frac{\lambda^{-2l+2p}}{3}\tau_3^{(2)}\right)$$

$$G_i = \int_{-h}^{h}\tau_i\alpha_3 d\alpha_3 = \frac{2h^2}{3}\lambda^{2p-c}\left(\tau_i^{(1)} + \lambda^{-2l+2p}\frac{3}{5}\tau_i^{(3)}\right)$$

$$S_{ij} = \int_{-h}^{h} \tau_{ij} d\alpha_3 = 2h\lambda^l \left(\tau_{ij}^{(0)} + \frac{\lambda^{-2l+2p}}{3}\tau_{ij}^{(2)}\right)$$

$$H_{ij} = \int_{-h}^{h} \tau_{ij}\alpha_3 d\alpha_3 = \frac{2h^2}{3}\lambda^{2p-c}\left(\tau_{ij}^{(1)} + \lambda^{-2l+2p}\frac{3}{5}\tau_{ij}^{(3)}\right)$$

$$N_i = \int_{-h}^{h} \tau_{i3} d\alpha_3 = 2h\lambda^p\left(\tau_{i3}^{(0)} + \frac{\lambda^{-l+2p-c}}{3}\tau_{i3}^{(2)} + \frac{\lambda^{-3l+4p-c}}{5}\tau_{i3}^{(4)}\right)$$

Transforming the system (1.10), (1.11) in combination with (1.12), we obtain the equilibrium equations of shell theory which agree with the standard equations and refined elasticity relationships. For brevity, let us write them in tensor form

$$T^{ik} = BE^{ik\alpha\beta}\varepsilon_{\alpha\beta} + (1+\nu) DF^{ik\alpha\beta}\mu_{\alpha\beta} + DP^{ik}\mu_\gamma{}^\gamma + \tag{1.13}$$

$$D\frac{\nu}{2(1-\nu)}G^{ik\alpha\beta}\nabla_\alpha\nabla_\beta\varepsilon_\gamma{}^\gamma + \frac{h^2}{3(1-\nu)}K^{ik\alpha\beta}\nabla_\alpha(q_\beta{}^+ + q_\beta{}^-) + \frac{\nu}{1-\nu}ha^{ik}(q_3{}^+ - q_3{}^-)$$

$$M^{ik} = D\left(E^{ik\alpha\beta} - \frac{1+\nu}{2}c^{\alpha\beta}c^{ik}\right)\mu_{\alpha\beta} + DH^{ik\alpha\beta}\varepsilon_{\alpha\beta} +$$

$$D\frac{\nu}{1-\nu}E^{ik\alpha\beta}b_{\alpha\beta}\varepsilon_\gamma{}^\gamma + D\frac{\nu h^2}{10(1-\nu)}G^{ik\alpha\beta}\nabla_\alpha\nabla_\beta\mu_\gamma{}^\gamma +$$

$$D\frac{4}{5}G^{ik\alpha\beta}\nabla_\alpha\gamma_{\beta 3} + \frac{h^3}{15}Q^{ik\alpha\beta}\nabla_\alpha(q_\beta{}^+ - q_\beta{}^-) + \frac{\nu h^2}{3(1-\nu)}a^{ik}(q_3{}^+ + q_3{}^-)$$

$$N^i = -\frac{2Eh}{3(1+\nu)}a^{i\alpha}\gamma_{\alpha 3} - \frac{h}{3}a^{i\alpha}(q_\alpha{}^+ - q_\alpha{}^-)$$

Here

$$\gamma_{\alpha 3} = -\nabla_\alpha w + u_\beta b_\alpha{}^\beta - c_\alpha{}^\beta u_{\beta 1} \tag{1.14}$$

$$E^{ik\alpha\beta} = a^{i\alpha}a^{k\beta} + \nu c^{i\alpha}c^{k\beta} \tag{1.15}$$

$$P^{ik} = \frac{\nu}{2(1-\nu)}[(1+2\nu)a^{i\alpha}a^{k\beta} + (2+\nu)c^{i\alpha}c^{k\beta}]\, b_{\alpha\beta}$$

$$F^{ik\alpha\beta} = 2Ha^{i\alpha}a^{k\beta} + 2a^{i\alpha}b^{k\beta} - \frac{1+3\nu}{2(1+\nu)}(a^{i\alpha}b^{k\beta} - b^{i\alpha}a^{k\beta})$$

$$V^{ik\alpha\beta} = a^{i\alpha}a^{k\beta} + c^{i\alpha}c^{k\beta} - c^{\alpha\beta}c^{ik}$$

$$G^{ik\alpha\beta} = E^{ik\alpha\beta} - \frac{1+\nu}{2}c^{\alpha\beta}c^{ik}$$

$$H^{ik\alpha\beta} = 2Ha^{i\alpha}a^{k\beta} + 2a^{i\alpha}b^{k\beta} - \frac{1+\nu}{2}(a^{i\alpha}b^{k\beta} - b^{i\alpha}a^{k\beta})$$

$$Q^{ik\alpha\beta} = \frac{2}{1-\nu}G^{ik\alpha\beta} + \frac{2\nu}{3(1-\nu)}V^{ik\alpha\beta}, \quad 2H = -b_\alpha{}^\alpha$$

$$K^{ik\alpha\beta} = \nu V^{ik\alpha\beta} + G^{ik\alpha\beta}, \quad B = \frac{2Eh}{1-\nu^2}, \quad D = \frac{2Eh^3}{3(1-\nu^2)}$$

The physical components of the tensors T^{ik}, M^{ik}, $\varepsilon_{\alpha\beta}$, $\mu_{\alpha\beta}$ are related to the stress resultants, moments, and strain components as follows:

$$[\varepsilon_{ii}] = \varepsilon_i, \quad [\varepsilon_{ij}] = \frac{1}{2}\omega, \quad [\mu_{ii}] = \varkappa_i, \quad [\mu_{ij}] = \tau - \frac{\omega}{2R_j} \tag{1.16}$$

$$[T^{ii}] = T_i, \quad [T^{ij}] = S_{ij}, \quad [M^{ii}] = G_i, \quad [M^{ij}] = H_{ij}$$

The remaining notation in (1.13) – (1.15) are similar to those used in [1].

The theory constructed differs qualitatively from a Love-type theory. As in the Naghdi theory, an elasticity relationship for the transverse stress resultants appears therein. Moreover, the shear $\gamma_{\alpha 3}$ characterizing the change in angle between the tangent to the α-line and the normal to the middle surface, is introduced by (1.14), where the following displacements

$$u_{\alpha,1} = \lambda^p v_1^{(1)}, \qquad u_{\beta,1} = \lambda^p v_2^{(1)} \tag{1.17}$$

should be introduced in addition to the displacements of the middle surface in order to define $\gamma_{\alpha 3}$.

Let us note that the shear $\gamma_{\alpha 3}$ is considered zero in Love-type theories. This is equivalent to discarding τ_{i3}^* in the last formula in (1.5), which introduces an error of the order of h_*^{2-2t}. The order of the system of differential equations for the theory obtained is sixteen.

Note. The elasticity relationships obtained can be used to refine the analysis of the simple edge effect, which is analyzed in [5].

2. Let us write the Naghdi elasticity relationships in the customary notation (we speak of the first variant of the elasticity relationships Naghdi proposed, in which the transverse shear is taken into account but the hypothesis about conserving the normal element is assumed)

$$\begin{aligned} T^{ik} &= BE^{ik\alpha\beta}\varepsilon_{\alpha\beta} + DH^{ik\alpha\beta}M_{\alpha\beta} + DH^{ik\alpha\beta}\nabla_\beta\gamma_{\alpha 3} \\ M^{ik} &= DG^{ik\alpha\beta}\mu_{\alpha\beta} + DH^{ik\alpha\beta}\varepsilon_{\alpha\beta} + DG^{ik\alpha\beta}\nabla_\alpha\gamma_{\beta 3} \\ N^i &= -\frac{5Eha^{i\alpha}}{6(1+\nu)}\gamma_{\alpha 3} - \frac{h}{6}\,a^{i\alpha}(q_\alpha^+ + q_\alpha^-) - \frac{h^2}{3}\,c^{i\alpha}c^{\beta\lambda}b_{\alpha\beta}(q_\lambda^+ + q_\lambda^-) \end{aligned} \tag{2.1}$$

where $E^{ik\alpha\beta}$, $H^{ik\alpha\beta}$, $G^{ik\alpha\beta}$ are defined by (1.15).

We compare (1.13) and (2.1). For example, let us compare the elasticity relationships for the stress resultants T^{ik}. Naghdi retains the terms $DH^{ik\alpha\beta}\mu_{\alpha\beta}$ and discards terms of the same order: $(1+\nu)DF^{ik\alpha\beta}\mu_{\alpha\beta}$ and $DP^{ik}\mu_\gamma^\gamma$ ($H^{ik\alpha\beta}$ and $F^{ik\alpha\beta}$ are distinct). Moreover, there are no terms taking account of the influence of the load in the Naghdi elasticity relationships. Neglecting them, results in an error $O(h_*)$ in the elasticity relationships for the normal stress resultants and an error $O(h_*^{2-2t})$ in the elasticity relationships for the bending moments.

It is therefore impossible to recognize (2.1) as asymptotically consistent. Naghdi introduced terms with $\gamma_{\alpha 3}$ therein but left out not only terms of the same order of smallness but also even greater terms (terms with $(q_3^+ - q_3^-)$).

Let us examine an example which shows to what this can result in numbers. We take a hinge-supported closed circular cylindrical shell of radius r, thickness $2h$ and length l; its face surfaces are loaded as follows:

$$\tau_3|_{\alpha_3=h} = q\sin k\xi, \qquad \tau_3|_{\alpha_3=-h} = 0 \tag{2.2}$$

$$k = \pi r/l, \qquad 0 \leqslant \xi \leqslant \frac{l}{r} \tag{2.3}$$

The coordinate ξ-line is directed along the generatrix, and the line α_3 along the normal to the middle surface. We seek the solution of the axisymmetric problem posed as

$$u = a\cos k\xi, \qquad v = 0, \qquad w = c\sin k\xi \tag{2.4}$$

The boundary conditions are hence satisfied automatically. Let us compare the normal displacement w obtained by means of the theories (1.13) and (2.1), representing it in the form

$$w = w_0 + \frac{h}{r} W$$

Here w_0 is the displacement determined by a Love-type theory, and W is the correction to the Love-type theory found by a refined theory. Omitting the simple calculations reducing to the solution of algebraic equations, we write down the final results. For $k = 1$ (the index of the state of stress variation equals zero) we obtain

$$W_1 = -(1-\nu)\frac{rq}{2E}\sin\xi, \qquad W_2 = -\frac{rq}{2E}\sin\xi \tag{2.5}$$

Here and later W_1 denotes the correction W found by the theory (1.13) and W_2 is the Naghdi theory correction. In order to avoid writing down tedious formulas in the remaining examples, the displacements are evaluated for $\nu = 0.3$. For $k = h_*^{-1/2}$ we obtain

$$W_1 = -1.12\frac{rq}{2E}\sin h_*^{-1/2}\xi, \qquad W_2 = -1.07\frac{rq}{2E}\sin h_*^{-1/2}\xi \tag{2.6}$$

It is seen from (2.5) and (2.6) that the discrepancy between the theories under consideration is more substantial in the case $k = 1$, $\nu = 0{,}3$ with the load (2.2).

Let us load the same shell with tangential forces as follows:

$$\tau_{\alpha 3}|_{\alpha_3 = h} = p\cos k\xi, \qquad \tau_{\alpha 3}|_{\alpha_3 = -h} = 0 \tag{2.7}$$

Formulas (2.3) and (2.4) are retained.

A computation carried out shows that complete agreement between the displacements w found by both theories is obtained for the load (2.7) with $k = 1$. For $k = h_*^{-1/2}$ a substantial discrepancy is obtained

$$W_1 = -2.02\frac{\sqrt{hr}}{2E}\, p\sin h_*^{-1/2}\xi, \qquad W_2 = -0.191\frac{\sqrt{hr}}{2E}\, p\sin h_*^{-1/2}\xi \tag{2.8}$$

N o t e. Naghdi [3] constructs a theory there in which he dispenses with the hypothesis of conservation of the length of a normal element. Without analyzing the appropriate relationships here, we just note that they are also inconsistent from the elucidated viewpoint.

3. By an asymptotic method we obtain the elasticity relationships for shells slightly resistant to shear.

Let us introduce a quantity characterizing the ratio between the shear moduli G and G', where G is the shear modulus for surfaces parallel to the middle surface, and G' is the shear modulus for planes perpendicular to the middle surface

$$G / G' = \eta = h_*^{-a} \tag{3.1}$$

We insert (3.1) into (1.5). This will result in a change in only the right side of the last formula in (1.5) where the additional factor λ^{al} will appear.

In constructing a theory of shells slightly resistant to shear, we should limit ourselves to the following values of a:

$$0 < a < 2-2t \tag{3.2}$$

since it can be shown, if $a \geqslant 2-2t$, that the construction of a two-dimensional theory becomes impossible.

Omitting computations analogous to those made in Sect. 1, let us write the elasticity relationships for shells slightly resistant to shear

$$T^{ik} = BE^{ik\alpha\beta}\varepsilon_{\alpha\beta} + r\frac{h^2\eta}{3(1-\nu)}G^{ik\alpha\beta}\nabla_\alpha(q_\beta^+ + q_\beta^-) + \tag{3.3}$$

$$r\frac{\nu}{1-\nu}ha^{ik}(q_3^+ - q_3^-)$$

$$M^{ik} = DG^{ik\alpha\beta}\mu_{\alpha\beta} + \frac{4}{5}DG^{ik\alpha\beta}\nabla_\alpha\gamma_{\beta 3} + rDH^{ik\alpha\beta}\varepsilon_{\alpha\beta} + rDR^{ik}\varepsilon_\gamma^\gamma +$$

$$r\frac{2h^3\eta}{15(1-\nu)}G^{ik\alpha\beta}\nabla_\alpha(q_\beta^+ + q_\beta^-) + \frac{r\nu h^3}{3(1-\nu)}a^{ik}(q_3^+ + q_3^-)$$

$$N^i = -\frac{2Eh}{3(1+\nu)\eta}a^{i\alpha}\gamma_{\alpha 3} - r\frac{h}{3}a^{i\alpha}(q_\alpha^+ - q_\alpha^-)$$

The notation (1.15) introduced in Sect. 1 remains valid for the formulas written down.

Two versions of the elasticity relationships are combined in (3.3); by assuming $r = 0$, we obtain the first version of elasticity relationships to the accuracy of

$$\varepsilon = \begin{cases} O(h_* + h_*^{4-4t-2a}), & 0 \leqslant t \leqslant 1/2 \\ O(h_*^{2-2t} + h_*^{4-4t-2a}), & 1/2 < t < 1 \end{cases} \tag{3.4}$$

and for $r = 1$ the second version to the accuracy of

$$\varepsilon = O(h_*^{2-2t} + h_*^{4-4t-2a}), \quad 0 \leqslant t < 1 \tag{3.5}$$

It should be noted that for $r = 1$ the theory (3.3) plays the same part for shells resistant to shear as the theory [1] relative to the remaining Love-type theories; a more complex and higher order system of shell theory equations than Naghdi's is obtained in attempting to go beyond the accuracy (3.5).

Comparing the elasticity relationships (3.3) for $r = 0$ with the Naghdi elasticity relationships (2.1), let us note that they differ negligibly in some numerical coefficients in the second order terms; thus the coefficient is $^4/_5$ for the term $DG^{ik\alpha\beta}\nabla_\alpha\gamma_{\beta 3}$ in (3.3), while it is unity for the same term in the Naghdi theory; moreover, the coefficient is $^2/_3$ in the first term on the right in (3.3) for the transverse stress resultants in the elasticity relationships, while it is $^5/_6$ in the Naghdi relations. It hence follows that the Naghdi elasticity relationships assure an accuracy to quantities on the order of (3.4) in an analysis of shells weakly resistant to shear.

Let us present a table (Table 1) of errors of the different theories for shells weakly resistant to shear. We represent the errors as $h_*^\alpha + h_*^\beta$ or h_*^α, and we enter the values of α and β in the Table. The errors obtained in analyzing shells without edges and a zero index of variability of the state of stress are written in the first row. The errors obtained in an analysis by the method of separating the state of stress [5], are given in the second row, the errors obtained for the membrane state of stress $(t = 0)$ are given in the upper line of the row, and the errors in analyzing simple edge effects $(t = {}^1/_2)$ are presented in the lower line of the row. Errors for states of stress with high variability are written in the third row.

Let us clarify the domain of applicability of the Naghdi theory as a function of the variability index of the state of stress and the number a which characterizes the shell shear stiffness.

An analysis of the shells under consideration by a Love-type theory not subjected to improvements yields an error $O(h_* + h_*^{2-2t-a})$, while by modified theory [1] the error is $O(h_*^{2-2t-a})$. It is seen from these error estimates that Love-type theory assures an accuracy which agrees with the accuracy of the Naghdi theory in an alaysis of closed shells with zero index of variability for values of $a < 1$, which is even exceeded in the second case. For $a = 1$ the errors of all these theories agree. The Naghdi theory yields a lesser error for $1 < a < 2$ (see the first row in the Table).

Table 1

t	Theories						
	Love-type		[1]	Naghdi		(3.3), $r=1$	
	α	β	α	α	β	α	β
0	1	$2-a$	$2-a$	1	$4-2a$	2	$4-2a$
0	1	$^3/_2-a$	$^3/_2-a$	1	$^5/_2-2a$	$^3/_2$	$^5/_2-2a$
$^1/_2$	$1-a$	—	$1-a$	1	$2-2a$	1	$2-2a$
$[^1/_2, 1]$	$2-2t-a$	—	$2-2t-a$	$2-2t$	$4-4t-2a$	$2-2t$	$4-4t-2a$

If the state of stress admits of separation into membrane and simple edge effects, then by using an iteration method desribed in [5], we obtain that for $a \leqslant {}^1/_2$ an analysis of the membrane state of stress by Naghdi method has no advantages as compared with Love-type theory. But for simple edge effects and states of stress with high variability, the Naghdi theory assures higher accuracy than a Love-type theory (see the second and third rows in the Table).

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